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Superfields with Higher Spin Fermionic Coordinates

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We consider superspaces with fermionic coordinates belonging to representations of the Lorentz group other than spin 1/2. A general diagrammatic method is introduced which facilitates the identification of the various component fields comprising the superfields. Explicit examples of superfields with fermionic coordinates belonging to $(1, 1/2)+(1/2, 1)$ and $(3/2, 0)+(0, 3/2)$ representations of the Lorentz group are worked out in detail.

1. INTRODUCTION

To date, all supersymmetries (Fayet and Ferrara, 1977; Salam and Strathdee, 1978; Nieuwenhuizen, 1981) employed in field theory are based on spin-1/2 fermionic generators which belong to the $(1/2, 0)+(0, 1/2)$ representation of the Lorentz group. The use of fermionic generators belonging to spins greater than one-half is forbidden by the theorem of Haag, *et al.* (1975). In this paper the assumptions underlying the theorem of Haag *et al.* are relaxed and a consistent supersymmetry algebra with generators belonging to the $(1, 1/2)+(1/2, 1)$ representation of the Lorentz group is given. The starting point is the theorem of Coleman and Mandula (1967).

The Coleman–Mandula theorem states that for a symmetry to be a symmetry of the S-matrix, the generators (bosonic) of the symmetry must close on the four-vector P_μ, the tensor of rotations and boosts $M_{\mu\nu}$, and a set of scalar charges B_I of an internal symmetry group. This theorem is supplemented by the theorem of Haag *et al.*, which states that in a theory consisting of massive particles, the bosonic and fermionic generators must close on P_μ, $M_{\mu\nu}$, B_I, and a further set of charges (Haag *et al.*, 1975) for the fermionic generators. The theorem restricts the fermionic generators to

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the $(1/2, 0)+(0, 1/2)$ representations of the Lorentz group. The arguments leading to this restriction run as follows. A fermionic generator Q_S and its Hermitian conjugate $\bar{Q}_S$ belong to the same algebra and satisfy the anticommutation relation

$$\{Q_S, \bar{Q}_S\}=\sum_x a_x T^x_{2S} \tag{1.1}$$

where T^x are all possible covariant terms and a_x are coefficients restricted by the closure of the algebra. The principal assumption in the theorem is that all T^x in equation (1.1) belong to spin $2S$, since one is dealing with an anticommutator of Q_S and $\bar{Q}_S$. Since the Coleman–Mandula theorem requires that the anticommutator close on the generators of the Poincaré group and a set of scalar charges, it follows that $2S$ is either zero or one and hence the maximum value of S equals one-half.

It is the last assumption that we have relaxed. Our criterion of an admissible fermionic generator to be a generator of the symmetry of the S-matrix is that its anticommutator must close on terms that respect covariance and the symmetry properties of the anticommutator. The T^x are all proportional to P_μ and the rest of the symmetry resides in the a_x. We have demonstrated this explicitly in the case of vector spinor generators $Q_{\mu\alpha}$ belonging to the $(1, 1/2)+(1/2, 1)$ representation of the Lorentz group (Pilot and Rajpoot, 1989). It now remains to show that this algebra leads to a superspace (Salam and Strathdee, 1974*a*,*b*; Ferrara *et al.*, 1974), and, in particular, to determine which component fields are represented in the multiplet.

There are at least two ways to find the particle content in a higher-spin supersymmetry theory. One involves the Wigner method of induced representations (Salam and Strathdee, 1974*a*,*b*; Ferrara *et al.*, 1974, 1981). It is a straightforward method, but does not allow for generalizations to other higher-spin supersymmetries if the algebra is not given *a priori*. The second method is based on a θ expansion; this approach does not require *a priori* knowledge of the algebra and can be universally applied if a higher-spin supersymmetry is otherwise known to be consistent. We shall concentrate on the second approach and determine the component fields for a spin $(1, 1/2)+(1/2, 1)$ superfield. We will then also apply this approach to the $(3/2, 0)+(0, 3/2)$ case. Although the latter supersymmetry has yet to be shown to exist, we proceed as if its existence had already been demonstrated. We wish to illustrate a method with a minimum number of indices and the second example lends itself readily to a demonstration of this sort. We emphasize, however, that any results based on the second example is a purely mathematical exercise at this point—nothing more.

The organization of this paper is as follows. In Section 2 we develop the superspace formalism for the $(1, 1/2)+(1/2, 1)$ case explicitly. The algebra is known, i.e., has been worked out, and the steps leading to the left- and right-handed chiral superfields are given. In Section 3 the expansion in θ variables is determined for this particular case using diagrammatic arguments. Finally, in Section 4, we assume a valid superspace exists for the $(3/2, 0)+(0, 3/2)$ case and apply the above-mentioned method to this example also. From these examples it should be clear how to generalize the diagrammatic method to any arbitrary higher-spin superfield.

2. (1, 1/2) + (1/2, 1) SUPERFIELD

Let $Q_{a\alpha}$ represent the $(1, 1/2)+(1/2, 1)$ fermionic, i.e., supersymmetric, generator; P_a, the four-dimensional translation generator; and M_{ab}, the four-dimensional rotation generator. The algebra which has been shown to close under the Jacobi identities is as follows:

$$[M_{ab}, M_{cd}] = -i(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}) \tag{2.1}$$

$$[M_{ab}, P_c] = -i(\eta_{ac}P_b - \eta_{bc}P_a) \tag{2.2}$$

$$[M_{ab}, Q_{c\alpha}] = (-i/2)(\gamma_{ab}Q_c)_\alpha - i(\eta_{ac}Q_{b\alpha} - \eta_{bc}Q_{a\alpha}) \tag{2.3}$$

$$[M_{ab}, \bar{Q}^{\alpha}_{c}] = (+i/2)(\bar{Q}_c\gamma_{ab})^\alpha - i(\eta_{ac}\bar{Q}^{\alpha}_{b} - \eta_{bc}\bar{Q}^{\alpha}_{a}) \tag{2.4}$$

$$[P_a, P_b] = 0 \tag{2.5}$$

$$[P_a, Q_{b\alpha}] = 0 = [P_a, \bar{Q}^{\alpha}_{b}] \tag{2.6}$$

$$\{Q_{a\alpha}, \bar{Q}^{\beta}_{b}\} = a[\eta_{ab}\not{P}_\alpha{}^\beta - \tfrac{1}{5}(\gamma_a P_b + \gamma_b P_a)_\alpha{}^\beta + (3i/5)\,\varepsilon_{abcd}(\gamma^c\gamma_5)_\alpha{}^\beta P^d] \tag{2.7}$$

The $Q_{a\alpha}$ is assumed to be Majoranic, i.e., $Q_{a\alpha}{}^C = (C\bar{Q}_a{}^T)_\alpha = Q_{a\alpha}$. Furthermore, being a $(1, 1/2)+(1/2, 1)$ charge, it is transverse in spinor space, i.e., obeys the identity $(\gamma^a Q_a)_\alpha = 0$.

Associated with the irreducible vector-spinor charge $Q_{a\alpha}$ are the vector-spinor coordinates $\theta_{a\alpha}$. The vector-spinor coordinates belong to the irreducible $(1, 1/2)+(1/2, 1)$ representation of the Lorentz group and are Majorana spinors,

$$\theta_{a\alpha} = (C\bar{\theta}^{T}_{a})_\alpha \tag{2.8}$$

Since the θ coordinates are Grassmannian, they obey the following relations:

$$\{\theta_{a\alpha}, \theta_{b\beta}\} = 0 = \{\theta_{a\alpha}, \bar{\theta}^{\beta}_{b}\} \tag{2.9}$$

The superspace is taken to consist of the four bosonic coordinates x_a and the 12 fermionic coordinates $\theta_{a\alpha}$. These coordinates can collectively be referred to as Z; a superfield on this superspace is denoted by $\Phi(Z)$ and is related to $\Phi(0)$ by exponentiation:

$$\Phi(Z)=\Phi(x_a,\theta_{a\alpha})=e^{x^aP_a}\,e^{\bar\theta_aQ^a}\Phi(0,0)$$
$$=e^{x^aP_a+\bar\theta^aQ_a}\Phi(0,0) \tag{2.10}$$

where we have used the commutivity of P_a with $Q_{a\alpha}$ to arrive at equation (2.10).

Next, we introduce a supersymmetric transformation operator acting on $\Phi(Z)$ which will take $\Phi(Z)$ to $\Phi(Z')$. One can write

$$e^{Z'}=e^{\bar\varepsilon^aQ_a}\,e^Z \tag{2.11}$$

where the $\varepsilon_{a\alpha}$ is an anticommuting, Majorana spinor parameter. As with any $(1,1/2)+(1/2,1)$ irreducible spinor, the general property $\gamma^a\varepsilon_a=0=\bar\varepsilon^a\gamma_a$ holds.

Equation (2.11) can be simplified using the algebra given in equations (2.1)–(2.7). Using equation (2.11) and the identity

$$e^A\,e^B=e^{A+B+[A,B]/2} \tag{2.12}$$

where A and B are arbitrary operators, one can show that

$$\exp(\bar\varepsilon^aQ_a)\exp Z=\exp\{\bar\varepsilon^aQ_a+x^aP_a+\bar\theta^aQ_a+\tfrac{1}{2}[\bar\varepsilon^aQ_a,\bar\theta^bQ_b]\} \tag{2.13}$$

Furthermore, the commutator on the right-hand side can be worked out using equation (2.7); one obtains after some Dirac algebra

$$[\bar\varepsilon^aQ_a,\bar\theta^bQ_b]=\bar\varepsilon^a\{Q_a,\bar Q_b\}\theta^b$$
$$=a[\bar\varepsilon^a\not P\theta_a+\tfrac{3}{5}i\varepsilon_{abcd}\bar\varepsilon^a\gamma^c\gamma_5\theta^bP^d] \tag{2.14}$$

where we have used the property that $\gamma^a\theta_a=0=\bar\varepsilon^a\gamma_a$.

The second term on the right-hand side of equation (2.14) can be further simplified by using the identity

$$\gamma_{abc}\equiv i\varepsilon_{abcd}\gamma^d\gamma_5$$
$$=\tfrac{1}{2}\{\gamma_{ab},\gamma_c\}=-\gamma_{bac}=+\gamma_{bca} \tag{2.15}$$

Again after some Dirac manipulation, a remarkable simplification results

$$\tfrac{3}{5}i\varepsilon_{abcd}\bar\varepsilon^a\gamma^c\gamma_5\theta^bP^d=-\tfrac{3}{10}iP^d\bar\varepsilon^a\{\gamma_{ab},\gamma_d\}\theta^b$$
$$=\tfrac{3}{10}(P^d\bar\varepsilon_b\gamma_d\theta^b+P^d\bar\varepsilon^a\gamma_d\theta_a)$$
$$=\tfrac{3}{5}\bar\varepsilon^a\not P\theta_a \tag{2.16}$$

By substituting the result of equation (2.16) back into equation (2.14), we obtain

$$[\bar{\varepsilon}^a Q_a, \bar{\theta}^b Q_b] = \tfrac{8}{5} a \bar{\varepsilon}^b \gamma_a \theta_b P^a \tag{2.17}$$

Therefore equation (2.13) now reads

$$\exp(\bar{\varepsilon}^a Q_a) \exp Z = \exp\{(x^a + \tfrac{1}{2}\bar{\varepsilon}^b \gamma^a \theta_b) P_a + (\bar{\theta}^a + \bar{\varepsilon}^a) Q_a\} \tag{2.18}$$

where the coefficient a has been set equal to unity without loss of generality.

A supersymmetrically transformed superfield can thus be expressed as

$$\Phi(Z') = e^{Z'} \Phi(0, 0) = \Phi(X^a + \tfrac{1}{2}\bar{\varepsilon}^b \gamma^a \theta_b, \bar{\theta}^a + \bar{\varepsilon}^a) \tag{2.19}$$

where we have allowed equation (2.18) to operate on $\Phi(0, 0)$. One sees that we have not only a translation in spinor space, but also an ordinary space-time translation of the amount $\frac{1}{2}\bar{\varepsilon}^b \gamma^a \theta_b$. Following customary procedure, we next expand (2.19) in a Taylor series about the point $(x^a, \bar{\theta}^a)$ to obtain

$$\Phi(Z') = \Phi(Z) + \bar{\varepsilon}^a d_a \Phi(Z) + \tfrac{1}{2}\bar{\varepsilon}^b \not{\partial} \theta_b \Phi(Z) \tag{2.20}$$

The spinor partial derivatives $d_{a\alpha}$ are defined as $d_{a\alpha} \equiv \partial / \partial \bar{\theta}^{a\alpha}$.

The action of a supersymmetric variation with infinitesimal parameter $\varepsilon_{a\alpha}$ can now be ascertained from equation (2.20). The result reads

$$\begin{aligned} \delta(\bar{\varepsilon}^a Q_a)\Phi(Z) &= \Phi(Z') - \Phi(Z) \\ &= (\bar{\varepsilon}^a d_a + \tfrac{1}{2}\bar{\varepsilon}^a \gamma^b \theta_a \partial_b)\Phi(Z) \end{aligned} \tag{2.21}$$

A supersymmetric covariant spinor derivative can be found using the above $(1, 1/2) + (1/2, 1)$ transformation law. It assumes the explicit form

$$D_{a\alpha} \equiv d_{a\alpha} - \tfrac{1}{2}(\not{\partial}\theta_a)_\alpha + \tfrac{1}{4}(\gamma_a \theta^b)_\alpha \partial_b \tag{2.22}$$

and one can easily demonstrate that the covariant derivative of a superfield transforms as the superfield itself, with the help of a little Dirac algebra.

We can also show that the anticommutator of (2.22) gives

$$\{D_{a\alpha}, D_{b\beta}\} = \not{\partial}_{\alpha\beta} \eta_{ab} - \tfrac{1}{4}(\gamma_a \partial_b + \gamma_b \partial_a)_{\alpha\beta} \tag{2.23}$$

Note that the result is symmetric under simultaneous interchange of $(a \leftrightarrow b)$ and $(\alpha \leftrightarrow \beta)$. From (2.23) it follows that

$$\bar{D}_a D_b - (a \leftrightarrow b) = 0 \tag{2.24}$$

Therefore

$$\bar{D}_a \begin{pmatrix} L \\ R \end{pmatrix} D_b - (a \leftrightarrow b) \equiv \bar{D}_a \left(\frac{1 \mp \gamma_5}{2} \right) D_b - (a \leftrightarrow b) = 0 \tag{2.25}$$

We allow this identity to act on an arbitrary superfield:

$$\bar{D}_a \begin{pmatrix} L \\ R \end{pmatrix} D_b\Phi - (a \leftrightarrow b) = 0 \tag{2.26}$$

From a study of electrodynamics, specifically how the vector potential follows from an antisymmetric field strength, it should be clear that the general solution to this equation is

$$\Phi_{aL} = L\Phi_a \equiv LD_a\Phi \tag{2.27}$$

Again the electromagnetic analogue is the vanishing of the field strength, an antisymmetric tensor, from which the general solution for the vector potential follows. We have demonstrated this for the left-handed superfields; the same can be demonstrated for the right-handed superfield. We see that instead of the ordinary left- and right-handed scalar chiral superfields in the $(1/2, 0) + (0, 1/2)$ case, we now have left- and right-handed *vector* chiral superfields for the $(1, 1/2) + (1/2, 1)$ case.

3. EXPANSION IN THE (1, 1/2) VARIABLE

In equation (2.8) we introduced the $(1, 1/2) + (1/2, 1)$ coordinate $\theta_{a\alpha}$ associated with $Q_{a\alpha}$. Like $Q_{a\alpha}$, it is transverse in spinor space, i.e., $\gamma^a\theta_a = 0$, and represents 12 components. The Majoranic $\theta_{a\alpha}$ can be written in $SL(2, C)$ Weyl spinor formalism as

$$\theta_{a\alpha} = \begin{pmatrix} \theta_{(AB\dot{C})} \\ \theta^{*(\dot{A}\dot{B}C)} \end{pmatrix} \tag{3.1}$$

where $\theta_{(AB\dot{C})}$ $(\theta^{*(\dot{A}\dot{B}C)})$ are the left (right)-handed chiral projections of $\theta_{a\alpha}$. In $SU(2) \times SU(2)$ notation these are the $(1, 1/2) + (1/2, 1)$ components, respectively. They are transformed into each other by parity transformations.

The $\theta_{a\alpha}$ are anticommuting coordinates, as seen from equation (2.9); written out in $SL(2, C)$ formalism, we claim that

$$\begin{aligned} \{\theta_{(AB\dot{C})}, \theta^{(DE\dot{F})}\} &= 0 \\ \{\theta^{*(\dot{A}\dot{B}C)}, \theta^{(DE\dot{F})}\} &= 0 \\ \{\theta_{(AB\dot{C})}, \theta^{*}{}_{(\dot{D}\dot{E}F)}\} &= 0 \\ \{\theta^{*(\dot{A}\dot{B}C)}, \theta^{*}{}_{(\dot{D}\dot{E}F)}\} &= 0 \end{aligned} \tag{3.2}$$

where the indices $A, B, \ldots = 1, 2$ and $\dot{A}, \dot{B}, \ldots = \dot{1}, \dot{2}$ are $SL(2, C)$ Weyl indices.

In ordinary spin $(1/2, 0) + (0, 1/2)$ SUSY an anticommuting θ_α implies that a superfield $\Phi(x, \theta)$ can be expanded in a finite power series in θ. The same holds for our new superfield $\Phi(X, \theta_a)$, where now the anticommuting $\theta_{a\alpha}$ must be used. To keep things as transparent as possible, we consider only a left-handed chiral multiplet where the expansion variable is $\theta_{(AB\dot{C})} = (L\theta_a)_\alpha$. This projection represents only six components.

To find which fields are represented in a (1, 1/2) supermultiplet, we first have to find those bilinear, trilinear, quartic, . . . , terms in $(L\theta_a)_\alpha = \theta_{(AB\dot{C})}$ that are irreducible. We expect for $(L\theta_a)$, $(L\theta_a)^2$, $(L\theta_a)^3, \ldots$ the following number of irreducible components. $(L\theta_a)^n$ gives $\binom{6}{n} = 6!\,[n!\,(6-n)!]^{-1}$ irreducible components. For example, $(L\theta_a)^4$ implies 15 irreducible component fields. $(L\theta_a)^7$ and higher powers vanish because of the nilpotency of spinors. There is no way to antisymmetrize successfully a six-component field seven times.

We now give the result, leaving the proof for the subsequent discussion. The result is

$$
\begin{aligned}
\Phi_L = {} & \phi_1 + \theta^{(AB\dot{C})}\varphi_{1(AB\dot{C})} + \theta^{(AB\dot{C})}\theta_{(AB\dot{C})}\phi_2 \\
& + \theta^{(AB\dot{E})}\theta^{(CD}{}_{\dot{E})}\phi_{2(ABCD)} + \theta^{(AE\dot{C})}\theta^{(B}{}_{E}{}^{\dot{D})}\phi_{2(AB\dot{C}\dot{D})} \\
& + \theta^{(DF\dot{A})}\theta_{(D}{}^{E\dot{B})}\theta_{(EF}{}^{\dot{C})}\varphi_{2(\dot{A}\dot{B}\dot{C})} \\
& + \theta^{(AF\dot{D})}\theta^{(BE}{}_{\dot{D})}\theta_{(EF}{}^{\dot{C})}\varphi_{2(AB\dot{C})} \\
& + \theta^{(AB\dot{G})}\theta^{(CF\dot{E})}\theta^{(D}{}_{F\dot{G})}\varphi_{2(ABCD\dot{E})} \\
& + \theta^{(AB\dot{E})}\theta_{(B}{}^{C\dot{F})}\theta_{(C}{}^{D}{}_{\dot{E})}\theta_{(DA\dot{F})}\phi_3 \\
& + \theta^{(AH\dot{E})}\theta^{(BF}{}_{\dot{E})}\theta^{(C}{}_{F}{}^{\dot{G})}\theta^{(D}{}_{H\dot{G})}\phi_{3(ABCD)} \\
& + \theta^{(AH\dot{E})}\theta^{(BF}{}_{\dot{E})}\theta_{(F}{}^{G\dot{C})}\theta_{(GH}{}^{\dot{D})}\phi_{3(AB\dot{C}\dot{D})} \\
& + \theta^{(AD\dot{H})}\theta^{(BE\dot{I})}\theta_{(D}{}^{F}{}_{\dot{I})}\theta_{(F}{}^{G\dot{C})}\theta_{(GE\dot{H})}\varphi_{3(AB\dot{C})} \\
& + \theta^{(A}{}_{F}{}^{\dot{A})}\theta^{(B}{}_{A}{}^{\dot{B})}\theta^{(C}{}_{B}{}^{\dot{C})}\theta^{(D}{}_{C\dot{A})}\theta^{(E}{}_{D\dot{B})}\theta^{(F}{}_{B\dot{C})}\phi_4
\end{aligned}
\tag{3.3}
$$

In the above ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are spin-(0, 0) scalars; $\varphi_{1(AB\dot{C})}$, $\varphi_{2(AB\dot{C})}$, and $\varphi_{3(AB\dot{C})}$ are spin-(1, 1/2) spinors: $\phi_{2(ABCD)}$ and $\phi_{3(ABCD)}$ are spin-(2, 0) conformal tensors; $\phi_{2(AB\dot{C}\dot{D})}$ and $\phi_{3(AB\dot{C}\dot{D})}$ are symmetric spin-(1, 1) tensors; $\varphi_{2(\dot{A}\dot{B}\dot{C})}$ is a spin-(0, 3/2) spinor; and, finally, $\varphi_{2(ABCD\dot{E})}$ is a spin-(2, 1/2) spinor. All tensors are designated by $\phi_{\ldots}$ and are characterized by an even number of symmetrized $SL(2, C)$ indices. All spinors are designated by $\varphi_{\ldots}$ and have an odd number of symmetrized Weyl indices. Even powers of $\theta_{(AB\dot{C})}$ in (3.3) always have bosonic components associated with them; an odd number will lead to fermionic components. As in any supersymmetric theory, the total number of fermionic components equals the total number of bosonic ones. In this case we have 32 + 32, giving a total of 64 components. The fields in expansion (3.3) can be given in the order of increasing θ as they occur in (3.3). We have in $SU(2)\times SU(2)$ notation

$$\begin{array}{l} \underset{1}{(0,0)}/\underset{6}{(1,1/2)}/(2,0)+\underset{5+9+1}{(1,1)}+(0,0)/\underset{10+6+4}{(2,1/2)+(1,1/2)+(0,3/2)} \\ \qquad \underset{5+9+1}{(2,0)+(1,1)+(0,0)}/\underset{6}{(1,1/2)}/\underset{1}{(0,0)} \end{array} \tag{3.4}$$

The number of components associated with each field is specified below the field in (3.4). In general in $SU(2)\times SU(2)$ notation, the total number of components represented by (j_1, j_2) is given by $(2j+1)$, where $|j_1-j_2|\leq j\leq(j_1+j_2)$. With (3.3) and (3.4) we have reproduced the results of previous work by means of a new but equivalent method.

We now prove that the expansion given by (3.3) is unique. We start with the bilinear form in $(L\theta_a)_\alpha$. Initially we have a total of 6×6 or 36 components for a bilinear form in $\theta_{(AB\dot{C})}$. They are given in $SU(2)\times SU(2)$ formalism as

$$(1,1/2)\times(1,1/2)=(2,1)+(2,0)+(1,1)+(1,0)+(0,1)+(0,0) \tag{3.5}$$

The first term on the right-hand side corresponds to zero contractions in $SL(2,C)$ indices between the θ's, the second to one dotted contraction, the third to one undotted contraction, the fourth to one dotted and one undotted contraction, etc., until no more contractions are possible.

Next we consider each of the products in turn. The first, (2, 1), implies a symmetrized product in $(AB\dot{C}DE\dot{F})$, but

$$\begin{aligned} \theta^{(AB\dot{C})}\theta^{(DE\dot{F})}\phi_{(AB\dot{C}DE\dot{F})} &= -\theta^{(DE\dot{F})}\theta^{(AB\dot{C})}\phi_{(AB\dot{C}DE\dot{F})} \\ &= -\theta^{(AB\dot{C})}\theta^{(DE\dot{F})}\phi_{(DE\dot{F}AB\dot{C})} \end{aligned} \tag{3.6}$$

In the first line we have used (3.2a); in the second we have relabeled indices. Clearly the vanishing result is because the indices are symmetrized for ϕ. [All irreducible fields in $SL(2, C)$ notation are characterized by fully symmetrized indices; otherwise they are reducible.]

The second product, (2, 0), implies one dotted contraction. We obtain

$$\begin{aligned} \theta^{(AB\dot{C})}\theta_{(A}{}^{E\dot{F})}\phi_{(BE\dot{C}\dot{F})} &= -\theta_{(A}{}^{E\dot{F})}\theta^{(AB\dot{C})}\phi_{(BE\dot{C}\dot{F})} \\ &= +\theta^{(AE\dot{F})}\theta_{(A}{}^{B\dot{C})}\phi_{(BE\dot{C}\dot{F})} \end{aligned} \tag{3.7}$$

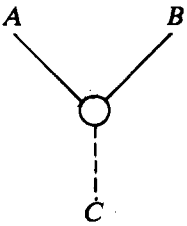
which does not vanish. The third product on the right-hand side of (3.5) does not vanish, for similar reasons. But the fourth gives

$$\begin{aligned} \theta^{(AB\dot{C})}\theta^{(D}{}_{B\dot{C})}\phi_{(AD)} &= -\theta^{(D}{}_{B\dot{E})}\theta^{(AB\dot{C})}\phi_{(AD)} \\ &= -\theta^{(DB\dot{E})}\theta^{(A}{}_{B\dot{C})}\phi_{(AD)}=0 \end{aligned} \tag{3.8}$$

Continuing this fashion demonstrates that only the bilinear forms contained in (3.3) are nonvanishing. Because of the antisymmetry of the $\theta_{(AB\dot{C})}$ upon interchange with another $\theta_{(DE\dot{F})}$, we note that all zero and *even-numbered* contractions must vanish.

We now associate with every $\theta_{(AB\dot{C})}$ a blob:

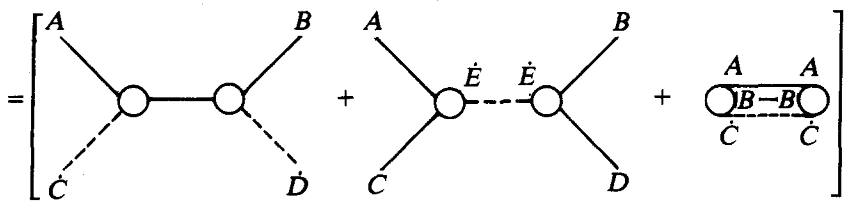
$\theta_{(AB\dot{C})} \triangleq$



(3.9)

where solid (dashed) lines represent undotted (dotted) indices. What kind of fully symmetrized Weyl field can be obtained if one contracts one $\theta_{(AB\dot{C})}$ with another $\theta_{(DE\dot{F})}$? The analysis above indicates that only an odd number of contractions between any two θ's (or blobs) is viable. The result is

$(1, 1/2) \times (1, 1/2)$

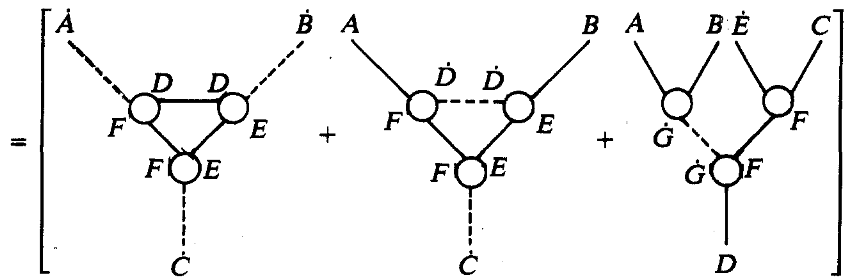


(3.10)

There are no others. At least one contraction is always necessary in order to antisymmetrize. Expansion (3.3) shows that these are exactly the represented bilinear terms in θ. The fields associated with these bilinear forms are characterized by the exposed indices in (3.10): $(AB\dot{C}\dot{D})$, $(ABCD)$, and (—).

We continue the analysis using this diagrammatic method. Consider the trilinear forms in $(L\theta_a)$. Antisymmetrizing gives

$(1, 1/2) \times (1, 1/2) \times (1, 1/2)$

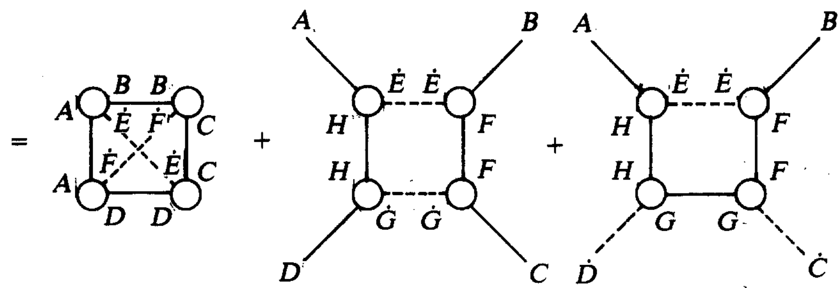


(3.11)

These are the only odd-numbered contractions which are possible between any two θ's. Also all θ's are necessarily contracted at least once in order to antisymmetrize. The exposed fully-symmetrized indices characterize the fields associated with these trilinear forms. These are given in equation (3.3) as the "coefficients" of the $(L\theta_a)^3$.

This line of development can be continued. The following results must hold. First the quartic terms give

$$(1, 1/2) \times (1, 1/2) \times (1, 1/2) \times (1, 1/2)$$

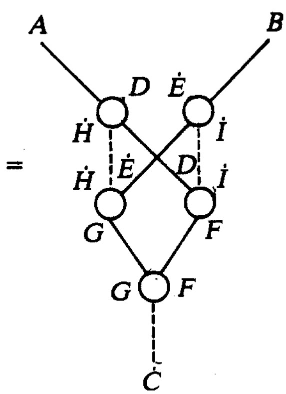


(3.12)

These are represented with their associated fields in the expansion given by equation (3.3). No two individual blobs or θ's are contracted more than twice with each other. There are no other quartic terms possible.

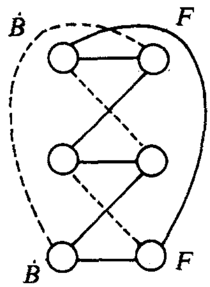
For $(L\theta_a)^5$ we obtain

$$(1, 1/2) \times (1, 1/2) \times (1, 1/2) \times (1, 1/2) \times (1, 1/2)$$



(3.13)

This term is given in (3.3) along with the $(L\theta_a)^6$ term. For the latter one obtains



(3.14)

This concludes the discussion of the expansion in $\theta_{(AB\dot{C})}$.

4. A (3/2, 0) + (0, 3/2) SUPERSPACE

We consider a superspace characterized by

$$Z_M = (x_a, \theta_{[ab]\alpha}) \tag{4.1}$$

where $\theta_{[ab]\alpha}$ is a (3/2, 0) + (0, 3/2) antisymmetric tensor–spinor coordinate. Associated with this coordinate will be an antisymmetric tensor–spinor generator $\theta_{[ab]\alpha}$ having the same irreducible components.

It is known that an arbitrary antisymmetric tensor–spinor

$$\Omega_{[ab]\alpha} = -\Omega_{[ba]\alpha}$$

can be uniquely decomposed into irreducible components under $SO(1, 3)$. The $\Omega_{[ab]\alpha}$ could represent coordinates, generators, fields, etc. Let us define

$$\begin{aligned} \Omega &\equiv (\gamma^{ab}\Omega_{ab}) \\ \Omega_a &\equiv (\gamma^{cd}\gamma_a\Omega_{cd}) \\ \tilde{\Omega}_{ab} &\equiv (\gamma_{ab}\gamma^{cd} + 3\gamma^{cd}\gamma_{ab})\Omega_{cd} \end{aligned} \tag{4.2}$$

These will be shown to be, respectively, the spin-(1/2, 0) + (0, 1/2), the spin-(1, 1/2) + (1/2, 1), and the spin-(3/2, 0) + (0, 3/2) components in $SU(2) \times SU(2)$ notation. We have in (4.2a)–(4.2c) 4 + 12 + 8 irreducible components. Relation (4.2b) obeys $\gamma^a\Omega_a = 0$, while (4.2c) satisfies

$$\begin{aligned} \gamma^a\tilde{\Omega}_{ab} &= 0 = \gamma^{ab}\tilde{\Omega}_{ab} \\ \gamma_a{}^c\tilde{\Omega}_{cb} &= -\tilde{\Omega}_{ab} \end{aligned} \tag{4.3}$$

The γ_{ab} is defined as $1/2[\gamma_a, \gamma_b]$, so that $\gamma_a\gamma_b = \eta_{ab} + \gamma_{ab}$, in general.

In Weyl $SL(2, C)$ formalism, the irreducible components represented by (4.2a)–(4.2c) are

$$\Omega_A = (L\Omega)_\alpha = (1/2, 0) \tag{4.4a}$$

$$\Omega^{*\dot{A}} = (R\Omega)_\alpha = (0, 1/2) \tag{4.4b}$$

$$\Omega_{(A\dot{A}B)} = (L\Omega_a)_\beta = (1, 1/2) \tag{4.5a}$$

$$\Omega^{*(\dot{A}A\dot{B})} = (R\Omega_a)_\beta = (1/2, 1) \tag{4.5b}$$

and

$$\tilde{\Omega}_{(ABC)} = (L\tilde{\Omega}_{ab})_\alpha = (3/2, 0) \tag{4.6a}$$

$$\tilde{\Omega}^{*(\dot{A}\dot{B}\dot{C})} = (R\tilde{\Omega}_{ab})_\alpha = (0, 3/2) \tag{4.6b}$$

We have assumed that the Ω's are Majoranic.

The decomposition (4.2) is based on the identity

$$\begin{aligned}\delta^{cd}_{ab}\delta_\alpha{}^\beta &\equiv \tfrac{1}{2}(\delta_a{}^c\delta_b{}^d - \delta^d_a\delta_b{}^c)\delta_\alpha{}^\beta \\ &= -\tfrac{1}{12}(\gamma_{ab}\gamma^{cd})_\alpha{}^\beta + \tfrac{1}{4}(\gamma^c\gamma_{ab}\gamma^d - \gamma^d\gamma_{ab}\gamma^c)_\alpha{}^\beta \\ &\quad - \tfrac{1}{24}(\gamma_{ab}\gamma^{cd} + 3\gamma^{cd}\gamma_{ab})_\alpha{}^\beta\end{aligned} \tag{4.7}$$

We note that the set of indices "*ab*" can be reversed with the set of indices "*cd*" within the second bracketed term on the right-hand side of (4.7). This will not change the result. Using (4.7) with (4.2), it is easily shown that

$$\Omega_{[ab]\alpha} = -\tfrac{1}{12}(\gamma_{ab}\Omega)_\alpha + \tfrac{1}{4}(\gamma_a\Omega_b - \gamma_b\Omega_a)_\alpha - \tfrac{1}{24}\tilde{\Omega}_{ab\alpha} \tag{4.8}$$

This decomposition (4.8) is unique. Defining the projection operators

$$\begin{aligned}(P_1)_{ab}{}^{cd} &\equiv -\tfrac{1}{12}(\gamma_{ab}\gamma^{cd}) \\ (P_2)_{ab}{}^{cd} &\equiv +\tfrac{1}{4}(\gamma^c\gamma_{ab}\gamma^d - \gamma^d\gamma_{ab}\gamma^c) \\ (P_3)_{ab}{}^{cd} &\equiv -\tfrac{1}{24}(\gamma_{ab}\gamma^{cd} + 3\gamma^{cd}\gamma_{ab})\end{aligned} \tag{4.9}$$

then we can write (4.8) as

$$\Omega_{ab} = (P_1\Omega)_{ab} + (P_2\Omega)_{ab} + (P_3\Omega)_{ab} \tag{4.10}$$

or $P_1 + P_2 + P_3 = 1$. Furthermore, since

$$P_iP_j = \delta_{ij}P_j, \qquad i, j = 1, 2, 3 \tag{4.11}$$

then the P's form a complete, normalized, and orthogonal decomposition. QED

Going back to equation (4.1), the $\theta_{[ab]\alpha}$ is to represent

$$(3/2, 0) + (0, 3/2)$$

components. [The $(1, 1/2) + (1/2, 1)$ components were treated previously, so a superspace based on these coordinates need not be considered here.] Properly speaking, the $\theta_{[ab]\alpha}$ in (4.1) is defined by a relation analogous to (4.2c); we have dropped the tilde for simplicity. In what follows, we are dealing exclusively with $(3/2, 0) + (0, 3/2)$ coordinates. Also, relations similar to (4.3) must hold for our new irreducible coordinates. The $\theta_{[ab]\alpha}$ anticommute, being true spinors [odd number of symmetrized $SL(2, C)$ indices]. Hence, $\{\theta_{[ab]\alpha}, \theta_{[cd]\beta}\} = 0$, or in $SL(2, C)$ notation

$$\begin{aligned} \{\theta_{(ABC)}, \theta^{(DEF)}\} &= 0, \qquad \{\theta^{*(\dot{A}\dot{B}\dot{C})}, \theta^{(DEF)}\} = 0 \\ \{\theta_{(ABC)}, \theta^{*}{}_{(\dot{D}\dot{E}\dot{F})}\} &= 0, \qquad \{\theta^{*(\dot{A}\dot{B}\dot{C})}, \theta^{*}{}_{(\dot{D}\dot{E}\dot{F})}\} = 0 \end{aligned} \tag{4.12}$$

The anticommuting θ's will guarantee a finite number of terms in an expansion about θ.

5. EXPANSION IN $\theta^{(ABC)}$

We wish to generate a superfield $\Phi(x_a, \theta_{(ABC)})$ based on an expansion in $\theta_{(ABC)}$. To keep things as simple as possible we consider only a left-handed $(3/2, 0)$ chiral superfield. Remember that $(L\theta_{[ab]})$ equals $\theta_{(ABC)}$ by relation (4.6).

Proceeding analogously to the $(1, 1/2)$ case, we seek those bilinear, trilinear, quartic, etc., terms in $\theta_{(ABC)}$ that are irreducible. We expect for $\theta_{(ABC)}$, $\theta^2_{(ABC)}$, $\theta^3_{(ABC)}$, etc., the following number of irreducible components: $(\theta_{(ABC)})^n$ gives $\binom{4}{n} = 4!\,[n!\,(4-n)!]^{-1}$ irreducible components. In all, we therefore expect $1+4+6+4+1$ or 16 components. The $\theta_{(ABC)}$ represent four independent components; written out these are $\theta_{(111)}$, $\theta_{(112)}$, $\theta_{(122)}$, and $\theta_{(222)}$. These can only be antisymmetrized a certain way, depending on the power of θ, because of the anticommutator (4.12a).

The result of this antisymmetrization in $\theta_{(ABC)}$ is given next, with the proof in the subsequent discussion. A $(3/2, 0)$ chiral superfield can be expanded as

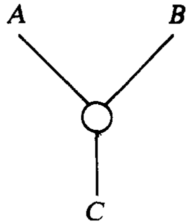
$$\begin{aligned} \Phi_L(x_a, \theta_{(ABC)}) = {} & \phi_1 + \theta^{(ABC)}\varphi_{1(ABC)} + \theta^{(ABC)}\theta_{(ABC)}\phi_2 \\ & + \theta^{(ABE)}\theta_{(E}{}^{CD)}\phi_{2(ABCD)} \\ & + \theta^{(ADE)}\theta^{(B}{}_{D}{}^{F)}\theta^{(C}{}_{EF)}\varphi_{2(ABC)} \\ & + \theta^{(ABC)}\theta_{(A}{}^{DE)}\theta_{(BD}{}^{F)}\theta_{(CEF)}\phi_3 \end{aligned} \tag{5.1}$$

where the ϕ's are bosonic components (even number of θ's) and the φ's are fermionic components (odd number of θ's). In $SU(2)\times SU(2)$ notation the fields given by increasing powers of $\theta_{(ABC)}$ in (5.1) are

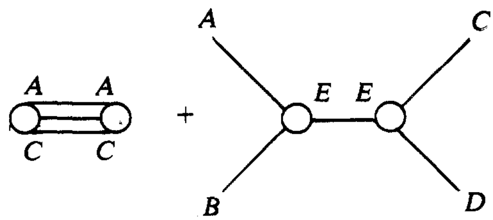
$$\begin{array}{ccccc}(0,0)/&(3/2,0)/&(2,0)+(0,0)/&(3/2,0)/&(0,0)\\ 1&4&5+1&4&1\end{array} \tag{5.2}$$

We have three scalars, two spin-(3/2, 0) fields, and one (2, 0) conformal tensor field. In (5.2) we have also specified the number of irreducible components associated with each field by the numbers given below the fields.

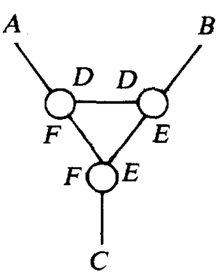
We prove equation (5.1) using the diagrammatic formalism developed in Section 2. We associate with each $\theta_{(ABC)}$ a blob:

$\theta_{(ABC)} \triangleq$  (5.3)

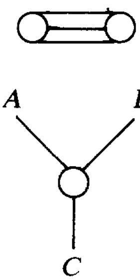
where each solid line represents an undotted $SL(2, C)$ index. Only undotted indices need be considered here. Now consider the various products of θ, keeping in mind that only an odd number of contractions are possible. An even-numbered contraction will always lead to a vanishing result because of the anticommuting nature of the θ's. For a bilinear term in θ,

$(3/2, 0)\times(3/2, 0) =$  (5.4)

These are the only possibilities and lead to the $(0, 0)+(2, 0)$ fields, respectively, given in (5.1). The trilinear term in θ gives

$(3/2, 0)\times(3/2, 0)\times(3/2, 0) =$  (5.5)

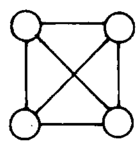
This is the only possibility and renders the $\varphi_{2(ABC)}$ field in (5.1). No two individual θ's can be contracted twice or more with respect to each other. A term such as



(5.6)

is inadmissible because the third θ is not antisymmetrized, i.e., contracted. Finally, the quartic term gives

$$(3/2,0)\times(3/2,0)\times(3/2,0)\times(3/2,0)= \tag{5.7}$$



This is the only possibility and leads to the ϕ_3 field given in (5.1).

We still have to give the parity content for the fields introduced in a spin-(3/2, 0) superfield expansion. Under parity the spin fields given by (5.2) transform naively as follows:

$$\begin{aligned}(j_1,j_2) &\to (-)^{(j_1-j_2)}(j_2,j_1)\\ (j,j) &\to \pm(j,j)\end{aligned} \tag{5.8}$$

where the + (−) refers to the ordinary (pseudo) tensor. One might naively expect $(0,0)^{\pm}$, $(3/2,0)^{\pm i}$, $(2,0)^{\pm}+(0,0)^{\pm}$, $(3/2,0)^{\pm i}$, and $(0,0)^{\pm}$, where the superscripts refer to the factor given in front of the right-hand side of (5.8). The θ's, however, twist the parity content because they are fermionic. Applying the parity operator twice in succession on a spinor gives minus one, i.e., a revolution of 4π is necessary to bring the spinor back into itself. Since the superfield has to remain a scalar or pseudoscalar, a twisting of the θ's, must imply an untwisting of the component fields, the "coefficients" of the θ's. This gives

$$(0,0)^{\pm},\quad (3/2,0)^{\pm i},\quad (2,0)^{\mp}+(0,0)^{\mp},\quad (3/2,0)^{\mp i},\quad (0,0)^{\mp} \tag{5.9}$$

as the spin/parity content for the (3/2, 0) superfield. The spin/parity content for the (1, 1/2) superfield (an expansion about $\theta_{(AB\dot{C})}$) need not be reproduced here. It has been worked out in a previous paper (Pilot and Rajpoot, 1988) using more formal arguments.

Summarizing, it should become apparent that any higher-spin fermionic coordinate, e.g., a (2, 1/2) $\theta_{(ABCD\dot{E})}$ or a (5/2, 0) $\theta_{(ABCDE)}$, will give a *finite* power series expansion in θ. Furthermore, the component fields associated with these superfield expansions are readily found using the diagrammatic (blob) technique.

Perhaps these added fermionic symmetries will improve on the renormalizability of ordinary gravity. For grand unified theories the increased particle content may prove interesting.

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